# From Black-Scholes formula, to local times and last passage times for certain submartingales

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February 28, 2008

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**Abstract:** Motivated by an expression of the standard Black-Scholes formula as (a multiple of) the cumulative function of a certain distribution on $\mathbb{R}_+$, we discuss a general extension of this identity starting from a general positive martingale converging to 0 as $t \to \infty$.

**Keywords:** Black-Scholes formula, last passage times, Azéma supermartingales.

# 1 Introduction and main results

• The origin of this note is the following identity:

$$E\left[\left|\exp\left(B_t - \frac{t}{2}\right) - 1\right|\right] = 2P(4B_1^2 \leq t) \tag{1}$$

where $(B_t, t \geq 0)$ is a one dimensional Brownian motion with $B_0 = 0$.
The analysis developed in this paper began with an investigation of the impact of stochastic volatility on the price of contracts that pay the average daily absolute price difference. We thank Michael Qian for bringing this question to our attention. In the Black-Scholes context on making an analogy between volatility and time we are soon led to considering the integral of the left hand side of equation (1) with respect to a measure on time and with the help of equation (1) it is much simpler to work with the right hand side of this equation.

Throughout this paper, we shall discuss various approaches to, and extensions of, this formula; the eager reader may already have a look at Example 1, in Section 5.
• Since $E\left[\exp\left(B_t - \frac{t}{2}\right) - 1\right] = 0$, the identity (1), is equivalent to:

$$E\left[\left(\exp\left(B_t - \frac{t}{2}\right) - 1\right)^{\pm}\right] = P(4B_1^2 \leq t) \tag{2}$$

Now, anyone who has dealt, for one reason or another, with Black-Scholes formula, will start using it to check that the LHS of (2) is indeed equal to its RHS. This involves, essentially, one integration by parts.
• However, the point of the present note is to exhibit a general extension of the identity (2), which necessitates no explicit knowledge of distributions. Here is this general result

**Theorem 1.** *For any $\geq 0$ continuous local martingale $(M_t, t \geq 0)$, which converges to 0 as $t \to \infty$, there is the identity: for any $b \geq 0$,*

$$E[F_t\,(b - M_t)^+] = b\,E\left[F_t\,1_{(g_\infty^{(b)} \leq t)}\right], \tag{3}$$

*where $F_t$ is any set in $\mathcal{F}_t$, and $g_\infty^{(b)} = \sup\{s : M_s = b\}$. (We make the convention that $\sup(\emptyset) = 0$.)*

• Here are some comments about Theorem 1:
obviously, (3) extends (2) in several respects:

a) $\left(\exp\left(B_t - \frac{t}{2}\right)\ ,\ t \geq 0\right)$ is replaced by a general $\geq 0$ continuous local martingale $(M_t, t \geq 0)$, which converges to 0, as $t \to \infty$ (we write $\mathcal{L}_0^{(+)}$ for this set of local martingales);

b) there is the presence of the test function $F_t$, for every $t \geq 0$; in fact, even more generally, (3) holds with $t$ replaced by $T$, any $(\mathcal{F}_t)$ stopping time, and $F_t$ replaced by $F_T 1_{(T<\infty)}$, where $F_T$ is any set in $\mathcal{F}_T$.

c) the level 1 in (2) is now replaced by the more general level $b$.

- The remainder of the note is organized as follows:
  - in Section 2, we prove Theorem 1;
  - in Section 3, we derive from (3) a general formula for the law of $g_\infty^{(b)}$;
  - in Section 4, we discuss the representation of $P(g_\infty^{(b)} > t|\mathcal{F}_t)$, which follows from Theorem 1 as a ratio: $\left(\frac{N_t}{S_t}, t \geq 0\right)$, for another element $(N_t)$ of $\mathcal{L}_0^{(+)}$, and $S_t = \sup_{s \leq t} N_s$;
  - in Section 5, we discuss several examples;
  - in Section 6, we develop a similar discussion as in Theorem 1, but this time with $E[F_t(M_t - b)^+]$; a little more care is needed;
  - Section 7 concludes, with the raising of a general question of representation of certain submartingales.

# 2 Proof of Theorem 1

Clearly, formula (3) is equivalent to:

(4) $$P(g_\infty^{(b)} \leq t|\mathcal{F}_t) = \left(1 - \frac{M_t}{b}\right)^+$$

and this formula follows from the fact that:

$$(g_\infty^{(b)} < t) = \left(\sup_{u \geq t} M_u < b\right)$$

Formula (4) is an application to the local martingale $(M_{t+u}, u \geq 0)$, which also belongs to $\mathcal{L}_0^{(+)}$, of "Doob's maximal identity":
if $(\mu_t, t \geq 0)$ belongs to $\mathcal{L}_0^{(+)}$, then: $\sup_{t \geq 0} \mu_t \overset{\text{(law)}}{=} \frac{\mu_0}{U}$, where $U$ is uniform on $[0, 1]$, and independent from $\mu_0$.
This identity has been discussed "at large" in, e.g., [5] and [7].

## 3 The law of $g_\infty^{(b)}$

Clearly, Theorem 1 shows some connection between the law of $M_t$, say for fixed time $t$, and that of $g_\infty^{(b)}$. This is made precise in the following Theorem 2, for which we need some further hypothesis about $(M_t, t \geq 0)$. We now assume:

(i) for every $t > 0$, the law of the r.v. $M_t$ admits a density $(m_t(x),\, x \geq 0)$, and: $(t,x) \to m_t(x)$ may be chosen continuous on $(0,\infty)^2$;

(ii) $d\langle M\rangle_t = \sigma_t^2 dt$, and there exists a jointly continuous function:

$$(t,x) \to E[\sigma_t^2 | M_t = x] \qquad \text{on} \qquad (0,\infty)^2 \; .$$

Then, the following holds:

**Theorem 2.** *The law of $g_\infty^{(b)}$ is given by:*

$$(5) \qquad P(g_\infty^{(b)} \in dt) = \left(1 - \frac{a}{b}\right)^+ \varepsilon_0(dt) + \frac{1_{(t>0)}}{2b}\, E[\sigma_t^2 | M_t = b] m_t(b) dt$$

*where $a = M_0$.*

**Remarks.**
**1)** Assume $a = 1$. Then, as already mentioned in Section 2, the law of $(\sup_{t\geq 0} M_t)$ is that of $\frac{1}{U}$, that is: it is "universal" for all elements $M \in \mathcal{L}_0^{(+)}$, with $M_0 = 1$. This is easily explained by Dubins-Schwarz representation: $M_t = \beta_{\langle M\rangle_t}$, with $(\beta_u, u \leq T_0(\beta))$ a BM starting from 1, and considered up to $T_0(\beta) \equiv \langle M\rangle_\infty$.
More generally, the law of $(L_\infty^b(M), b \geq 0)$ is also "universal", since:

$$L_\infty^b(M) = L_{T_0(\beta)}^b(\beta)\, , \; (b \geq 0)$$

On the other hand, as shown by formula (3), the law of $g_\infty^{(b)}(M)$ depends on the law of $M \in \mathcal{L}_0^{(+)}$ (while, for the same reason as before, that of $\langle M\rangle_{g_\infty^{(b)}(M)}$ is universal...).
**2)** Formula (5) extends, in the framework of $\mathcal{L}_0^{(+)}$ a similar result for the laws of last passage times for a transient diffusion as obtained in Pitman-Yor [8].
**3)** Assume $a = b = 1$, for simplicity. We may be interested to obtain a quite general distribution on $\mathbb{R}_+$ for the law of $g_\infty^{(1)}$.
Recall our original example:

$$E\left[\left(1 - \exp\left(B_t - \frac{t}{2}\right)\right)^+\right] = P(4B_1^2 \leq t)$$

Thus, in order to obtain on the RHS the cumulative function $F(t)$ instead of $P(4B_1^2 \leq t)$, it suffices to look for an increasing function $(h(t), t \geq 0)$ such that:
(6)

$$F(t)(\equiv P(X \leq t)) = P(4B_1^2 \leq h(t)) \equiv P\left(|B_1| \leq \sqrt{\frac{h(t)}{4}}\right) \equiv \mathcal{N}\left(\sqrt{\frac{h(t)}{4}}\right)$$

where $\mathcal{N}(x) = \sqrt{\frac{2}{\pi}} \int_0^x dy\, e^{-\frac{y^2}{2}}$.
Now, if $F$ is continuous and strictly increasing, there is only one such function $h$, and:

$$h(X) \overset{\text{(law)}}{=} 4B_1^2$$

or, equivalently:

$$X \overset{\text{(law)}}{=} h^{-1}(4B_1^2)$$

In the general case, we may take, from (6):

$$\sqrt{\frac{h(t)}{4}} = \mathcal{N}^{-1}(F(t))$$

In our martingale framework, we consider:

$$\mathcal{E}_{h(t)} \equiv \exp\left(B_{h(t)} - \frac{h(t)}{2}\right)$$

in the filtration $(\mathcal{F}_{h(t)}, t \geq 0)$.
<u>To summarize:</u> For every continuous r.v. $X \geq 0$, i.e: $F_X(t)$ is continuous, there exists a filtered probability space and a continuous martingale $(M_t)$ in $\mathcal{L}_0^{(+)}$ such that:

$$E[F_t(1 - M_t)^+] = P(F_t\, 1_{g \leq t})$$

with $g \overset{\text{(law)}}{=} X$.
**Proof of Theorem 2.**
**i)** We first use Tanaka's formula to obtain

$$E[(b - M_t)^+] = (b - a)^+ + \frac{1}{2} E[L_t^b]$$

where $(L_t^b, t \geq 0)$ denotes the local time at level $b$ for $(M_t, t \geq 0)$.
Thus, from formula (3), there is the relationship:

$$P(g_\infty^{(b)} \in dt) = \left(1 - \frac{a}{b}\right)^+ \varepsilon_0(dt) + \frac{1_{(t>0)}}{2b}\, d_t(E[L_t^b]) \tag{7}$$

and formula (5) is now equivalent to the following expression for $d_t(E[L_t^b])$:

$$d_t(E[L_t^b]) = dt(E[\sigma_t^2|M_t = b])m_t(b) \qquad (t > 0) \tag{8}$$

**ii**) We now prove (8):
The density of occupation formula for the local martingale $(M_t)$ writes:
for every $f : \mathbb{R}_+ \to \mathbb{R}_+$, Borel,

$$\int_0^t ds\sigma_s^2\, f(M_s) = \int_0^\infty db\, f(b) L_t^b$$

Thus, taking expectations on both sides, we obtain:

$$E\left[\int_0^t ds\sigma_s^2\, f(M_s)\right] = \int_0^\infty db\, f(b) E[L_t^b]$$

The LHS equals:

$$\begin{aligned} & \int_0^t ds\, E[E(\sigma_s^2|M_s) f(M_s)] \\ = & \int_0^\infty db\, f(b) \int_0^t ds\, m_s(b)\, E[\sigma_s^2|M_s = b] \end{aligned}$$

and formula (8) now follows easily.

∎

# 4 Representations of Azéma supermartingales

A third point consists in making a connection between Theorem 1 and the representation of a large class of Azéma supermartingales

$$(P(L > t|\mathcal{F}_t), t \geq 0)$$

associated to the end $L$ of a previsible set $\Gamma$, ie: $L = \sup\{t : (t, \omega) \in \Gamma\}$, as discussed in Nikeghbali-Yor ([7],Theorem 4.1): the set up in that paper is that $L$ is the end of a previsible set (on a given filtered probability space) such that:

$$(CA) \qquad \begin{cases} a) & \text{all } (\mathcal{F}_t) \text{ martingales are continuous;} \\ b) & \text{for any stopping time } T,\ P(L = T) = 0\,. \end{cases}$$

($C$ stands for continuous, and $A$ for avoiding (stopping times)). Under $(CA)$, there exists a positive continuous local martingale $(N_t, t \geq 0)$ such that:

$$P(L > t|\mathcal{F}_t) = \frac{N_t}{S_t}\ , \tag{9}$$

where $S_t = \sup_{s \le t} N_s$ , $t \ge 0$. Let us now make the connection with Theorem 1; for simplicity, let us take $a = b = 1$. Clearly, as we already noted, the identity (3) is equivalent to:

$$P(g_\infty^{(1)} \le t | \mathcal{F}_t) = (1 - M_t)^+$$

or to:

$$P(g_\infty^{(1)} > t | \mathcal{F}_t) = M_t \wedge 1 \tag{10}$$

We shall now show that $(M_t \wedge 1)$ may be written in the form $\left( \frac{N_t}{S_t}, t \ge 0 \right)$.

**Theorem 3.** *There is the representation:*

$$(M_t \wedge 1) = \frac{N_t}{S_t} \tag{11}$$

*in a unique manner, where $(N_t)$ belongs to $\mathcal{L}_0^{(+)}$, and $S_t = \sup_{s \le t} N_s$ are given by:*

$$\begin{cases} N_t = (M_t \wedge 1) \exp\left( \frac{1}{2} L_t^{(1)} \right) & (12a) \\ S_t \equiv \sup_{s \le t} N_s = \exp\left( \dfrac{1}{2} L_t^{(1)} \right) & (12b) , \end{cases} \tag{12}$$

*and $(L_t^{(1)}, t \ge 0)$ is the local time at 1 for $(M_t, t \ge 0)$.*

**Proof of Theorem 3.**
**a)** From Tanaka's formula, we have (we still assume that $M_0 = 1$):

$$(M_t \wedge 1) = 1 + \int_0^t 1_{M_s < 1} dM_s - \frac{1}{2} L_t^{(1)} \tag{13}$$

**b)** On the other hand, Itô's formula yields:

$$\begin{aligned} \frac{N_t}{S_t} &= 1 + \int_0^t \frac{dN_s}{S_s} - \int_0^t \frac{N_s dS_s}{S_s^2} \\ &= 1 + \int_0^t \frac{dN_s}{S_s} - \int_0^t \frac{dS_s}{S_s} \\ &= 1 + \int_0^t \frac{dN_s}{S_s} - \log(S_t) \end{aligned} \tag{14}$$

Thus, comparing (13) and (14), in order that the identity (11) holds, we need both equalities:

$$\int_0^t \frac{dN_s}{S_s} = \int_0^t 1_{M_s < 1} dM_s \tag{15}$$

$$(16) \qquad \log(S_t) = \frac{1}{2} L_t^{(1)}$$

This necessitates that $(N_t)$ is given by (12a).
Conversely, as we start from formula (12a), we find out that:

i) $(N_t, t \geq 0)$, thus defined, is a local martingale,

ii) it converges to 0, as $t \to \infty$, since $L_\infty^{(1)} < \infty$, thus:

$$N_t \leq \left( \exp\left( \frac{1}{2} L_\infty^{(1)} \right) \right) M_t$$

iii) $S_t$ is given by $\exp\left(\frac{1}{2} L_t^{(1)}\right)$

■

**Remarks.**
**1)** We note that formula (11) expresses precisely the multiplicative decomposition of the supermartingale $(M_t \wedge 1)$ as the product of a local martingale, and a decreasing process $(1/S_t, t \geq 0)$.
**2)** Note that every Azéma supermartingale, under the condition $(CA)$ may be expressed in the form (9): $\left(\frac{N_t}{S_t}, t \geq 0\right)$. On the other hand, it is not true that every Azéma supermartingale may be expressed as $(M_t \wedge 1)$; these representations are more particular:
indeed, from formulae (15) and (16) we deduce that:

$$(17) \qquad d{<}N{>}_s = \exp(L_s^{(1)}) 1_{(M_s<1)} (d{<}M{>}_s)$$

Now, in a Brownian setting (to simplify), we have: $d{<}N{>}_s = n_s^2 ds$ and $d{<}M{>}_s = m_s^2 ds$, for two $(\mathcal{F}_s)$ previsible processes $(m_s^2)$ and $(n_s^2)$. Now, (17) implies: $n_s^2 = \exp(L_s^{(1)}) 1_{(M_s<1)} \, m_s^2$, $ds \, dP$ a.s.
Consequently, $n_s^2 = 0 \; ds \, dP$ a.s. on $\{(s,\omega) : M_s > 1\}$. However, this cannot be satisfied if we start from $N$ such that $n_s^2 > 0$, for all $s > 0$; note that the random set $\{s : M_s > 1\}$ is not empty; if it were, then the local time at 1 of $M$ would be 0, and $M$ would be identically equal to 1.

# 5 Some examples

In this Section, we propose a number of local martingales in $\mathcal{L}_0^{(+)}$, for which we show how to compute the law of $g_\infty^{(b)}$, either from direct arguments, or with the help of Theorem 2.

**Example 1:** *$M_t = \exp\left(B_t - \frac{t}{2}\right)$ where $(B_t, t \geq 0)$ is a one dimensional Brownian motion starting from 0.*

Note that:

$$g_\infty^{(1)} = \sup\left\{t : B_t - \frac{t}{2} = 0\right\} \ ;$$

Therefore, by time inversion:

$$\begin{aligned} \frac{1}{g_\infty^{(1)}} &\overset{\text{(law)}}{=} \inf\left\{u : B_u = \frac{1}{2}\right\} \\ &\overset{\text{(law)}}{=} \frac{1}{4B_1^2} \ , \end{aligned} \tag{18}$$

hence formula (2) is now explained simply.

Note that the same time inversion argument yields the following general result:

$$\begin{aligned} \frac{1}{g_\infty^{(b)}} &\overset{\text{(law)}}{=} \inf\left\{t : B_t - t(\log b) = \frac{1}{2}\right\} \\ &\overset{\text{(law)}}{=} \frac{1}{(\log b)^2} \inf\left\{u : B_u - u = \frac{(\log b)}{2}\right\} \end{aligned}$$

**Example 2:** *$M_t = B_{t \wedge T_0}$ where $(B_t, t \geq 0)$ is a one-dimensional Brownian motion starting from 1.*

Then, for $b < 1$, we can find simply the law of

$$g_\infty^{(b)} = \sup\{t < T_0 : B_t = b\}$$

By time reversal, $(B_{T_0 - u}, u \leq T_0)$ is a $BES(3)$ process $(R_u)$ starting from 0, considered up to its last passage time at 1. Thus, we have:

$$T_0 - g_\infty^{(b)} = \inf\{u : R_u = b\} \ ,$$

therefore:

$$e^{-\lambda} = \frac{\lambda b}{\sinh(\lambda b)} E\left[\exp\left(-\frac{\lambda^2}{2} g_\infty^{(b)}\right)\right] \ ,$$

ie:

$$\begin{aligned} E\left[\exp\left(-\frac{\lambda^2}{2} g_\infty^{(b)}\right)\right] &= \frac{e^{-\lambda(1-b)} - e^{-\lambda(1+b)}}{(2\lambda b)} \\ &= \frac{1}{2b} \int_{1-b}^{1+b} dx \ \exp(-\lambda x) \ , \end{aligned}$$

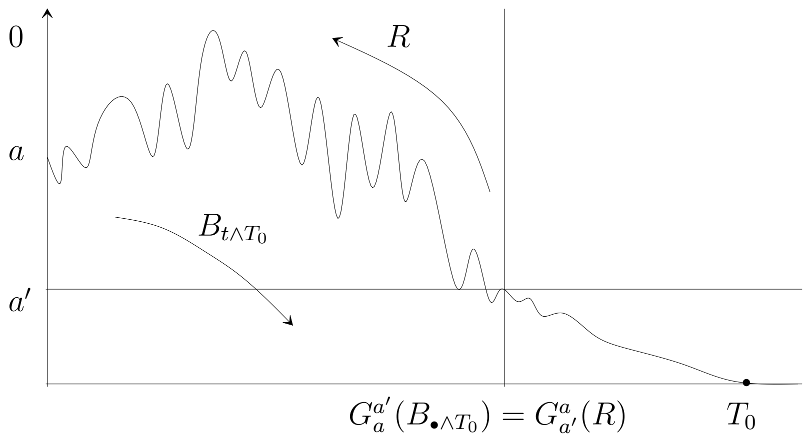
from which we deduce: $g_\infty^{(b)} \stackrel{\text{(law)}}{=} T_{U_b}$, where $T_a = \inf\{t : B_t^{(0)} = a\}$, and $U_b$ is independent from $B^{(0)}$, and uniformly distributed on $[1-b, 1+b]$.
Thus:

$$g_\infty^{(b)} \stackrel{\text{(law)}}{=} \frac{U_b^2}{B_1^2} \tag{19}$$

**Example 3:** $M_t = \frac{1}{R_t}$ *where* $(R_t, t \geq 0)$ *denotes a* $BES(3)$ *process starting from 1 (to simplify!).*
Note that $(M_t, t \geq 0)$ is a strict local martingale, but the previous discussion applies nonetheless.
Then, for $b < 1$, we have:

$$g_\infty^{(b)}(M) = \sup \left\{ t : \frac{1}{R_t} = b \right\} = \sup \left\{ t : R_t = \frac{1}{b} \right\}$$

Let us now consider the time reversal result between BM issued from $a > 0$, up to its first hitting time of 0, and $BES(3)$, up to its last passage time at $a$




We now apply this result with:

$$a' = 1\,;\ a = \frac{1}{b} > 1\,.$$

Let us describe precisely the law of $G_{a'}^a(R)$.
We have :

$$\exp(-\lambda a) = \left( \frac{\lambda a'}{\operatorname{sh}(\lambda a')} \right) E \left[ \exp\left( -\frac{\lambda^2}{2} G_{a'}^a(R) \right) \right]$$

Hence:

$$\begin{aligned} E\left[\exp\left(-\frac{\lambda^2}{2}G^a_{a'}(R)\right)\right] &= \frac{\mathrm{sh}(\lambda a')}{\lambda a'}\exp(-\lambda a) \\ &= \frac{1}{2(\lambda a')}(e^{\lambda a'} - e^{-\lambda a'})e^{-\lambda a} \\ &= \frac{1}{(2\lambda a')}\{e^{-\lambda(a-a')} - e^{-\lambda(a+a')}\} \\ &= \frac{1}{(2a')}\int_{(a-a')}^{a+a'} dx\, e^{-\lambda x} \end{aligned}$$

Hence:

$$\begin{aligned} G^a_{a'}(R) &\overset{\text{(law)}}{=} T_{(U_{[a-a',a+a']})} \\ &\overset{\text{(law)}}{=} \left(\frac{U^2_{[a-a',a+a']}}{B_1^2}\right) \end{aligned} \tag{20}$$

where $U_{[\alpha,\beta]}$ denotes a uniform variable on $[\alpha, \beta]$ independent of $B_1$.

We note that for the 3 previous examples, we did not use Theorem 2, as we were able to obtain directly the distribution of $g^{(b)}_\infty$. For the next examples, we make use of Theorem 2.

**Example 4:** $M_t = \cosh(B_t)\exp\left(-\frac{t}{2}\right)$ *where* $(B_t, t \geq 0)$ *is a one-dimensional Brownian motion starting from 0.*
From Itô's formula, we obtain:

$$\sigma_t = \sinh(B_t)e^{-\frac{t}{2}}$$

Hence $\sigma_t^2 = M_t^2 - e^{-t}$; therefore, formula (7) yields:

$$P(g^{(b)}_\infty \in dt) = \left(1 - \frac{1}{b}\right)^+ \varepsilon_0(dt) + \frac{1_{(t>0)}}{2b}(b^2 - e^{-t})m_t(b)dt$$

(we note that $m_t(b) = 0$, for $b^2 - e^{-t} \leq 0$).
We leave the (easy) computation of $m_t(b)$ to the reader.

**Example 5:** $M_t = \frac{1}{\sqrt{1-t}}\exp\left(-\frac{B_t^2}{2(1-t)}\right)$ , $t < 1$. This martingale is the Radon-Nikodym density on $\mathcal{F}_t = \sigma\{B_s, s \leq t\}, t < 1$, between the laws of the standard Brownian bridge and Brownian motion on the time interval

$[0,1]$, tends to 0 as $t \to 1-$.
Likewise, we obtain from Itô's formula:

$$\sigma_t^2 = \frac{B_t^2}{(1-t)^2} M_t^2 \equiv h(t, M_t)$$

for a certain function $h$. Again, we leave the details to the reader.

We note that **Examples 4 and 5** exhibit martingales $(M_t)$ which are inhomogeneous Markov processes. The interested reader may like to consult a list of results at the end of [3], where a number of Azéma supermartingales $P(L > t|\mathcal{F}_t)$, $t \geq 0$ are computed, in a Markovian framework.

# 6 From puts to calls: a little more care is needed

In pricing financial options, the left-hand side of (3) arises very naturally in terms of put options, e.g. when considering

$$E[(b - M_t)^+] \tag{21}$$

On the other hand, the price of a call option is:

$$E[(M_t - b)^+] \tag{22}$$

A most common argument to "reduce" (22) to (21) is to involve "call-put parity" and/or "change of numéraire". Mathematically, this means that we consider the new probability $Q$ defined via:

$$Q_{|\mathcal{F}_t} = M_t \bullet P_{|\mathcal{F}_t} \tag{23}$$

and the martingale $\left(\frac{1}{M_t}, t \geq 0\right)$ under $Q$, since:

$$\begin{aligned} E_p[(M_t - b)^+] &= E_Q\left[\left(1 - \frac{b}{M_t}\right)^+\right] \\ &= b\, E_Q\left[\left(\frac{1}{b} - \frac{1}{M_t}\right)^+\right] \end{aligned} \tag{24}$$

However, two difficulties arise in order to perform these operations rigorously:

i) in order that $Q$, as "defined" via (23), be a probability, we need that $(M_t, t \geq 0)$ is a true martingale under $P$, ie: it satisfies in particular $E_P(M_t) \equiv 1$;

ii) some care is needed also concerning (24); in particular $M_t$ could take the value 0 on some $\mathcal{F}_t$-set of positive $P$-probability.

To summarize, (23) and (24) are correct if $(M_t, t \geq 0)$ is a strictly positive true martingale under $P$.
Formally, this may be stated as:

**Proposition 4.** *If $(M_t)$ is a strictly positive true continuous martingale under $P$, define $P^M$ via: $P^M_{|\mathcal{F}_t} = M_t \bullet P_{|\mathcal{F}_t}$.*
*Denote: $g^{(1)}_\infty = \sup\{t \geq 0 : M_t = 1\}$. Then*

$$
\begin{aligned}
&i) \qquad E_P[F_t(M_t-1)^+] = E^M\left[F_t\, 1_{(g^{(1)}_\infty \leq t)}\right] \text{ , for every } F_t \in \mathcal{F}_t \\
&ii) \qquad E_P(F_t|M_t-1|) = E_P\left(F_t\, 1_{(g^{(1)}_\infty \leq t)}\right) + E^M\left(F_t\, 1_{(g^{(1)}_\infty \leq t)}\right) \\
&iii) \qquad g^{(1)}_\infty \text{ has the same distribution under } P \text{ and under } P^M
\end{aligned}
$$

**Proof of Proposition 4.**
**i)** We write:

$$
\begin{aligned}
E_P[F_t(M_t-1)^+] &= E^M\left[F_t\left(1-\frac{1}{M_t}\right)^+\right] \\
&= E^M\left[F_t\, 1_{(g^{(1)}_\infty \leq t)}\right]
\end{aligned}
$$

from Theorem 1, since $\left(\frac{1}{M_t}, t \geq 0\right)$ is, under $P^M$, a martingale which converges to 0 as $t \to \infty$.
**ii)**

$$E_P(F_t|M_t-1|) = E_P(F_t(M_t-1)^+) + E_P(F_t(1-M_t)^+)$$

and we apply both the previous result and Theorem 1.
**iii)** Taking $F_t = 1$ in i), we obtain:

$$
\begin{aligned}
P^M(g^{(1)}_\infty \leq t) &= E_P((M_t-1)^+) = E_P(M_t-1) + E_P((M_t-1)^-) \\
&= E_P((1-M_t)^+) \\
&= E_P(g^{(1)}_\infty \leq t) \qquad \text{(from Theorem 1)}
\end{aligned}
$$

■

**Remark 5.** *We note that Proposition 4 extends in the case when $(M_t)$ is a true continuous martingale, taking values in $\mathbb{R}_+$, but it may vanish, ie: $P(T_0 < \infty) > 0$.*
*Indeed, under $P^M$, $T_0 = \infty$ a.s., and the previous arguments are still valid.*

In order to obtain some analogue of Theorem 1 for

$$E[F_t(M_t - b)^+]$$

in the general case when $(M_t)$ is a local martingale belonging to $\mathcal{L}_0^{(+)}$, we shall proceed directly.
Before discussing in a general framework, we write some version of the identity (3) for $M_t = \frac{1}{R_t}, t \geq 0$, where $R$ is a $BES(3)$, starting from 1.
We denote by $W_a$ and $P_a^{(3)}$ $(a > 0)$ the respective laws of Brownian motion and $BES(3)$ starting at $a > 0$, on this canonical space $C(\mathbb{R}_+, \mathbb{R})$, with $X_t(\omega) = \omega(t)$, and $\mathcal{F}_t = \sigma\{X_s, s \leq t\}$; then, there is the well-known Doob $h$-transform relationship:

$$P^{(3)}_{a|\mathcal{F}_t} = \frac{X_{t\wedge T_0}}{a} \bullet W_{a|\mathcal{F}_t} \tag{25}$$

**Proposition 6.** *There is the identity*

$$E_1^{(3)}\left[F_t\left(\frac{1}{X_t} - 1\right)^+\right] = W_1\left(F_t\, 1_{(\gamma \leq t \leq T_0)}\right) \tag{26}$$

*for every $F_t$ in $\mathcal{F}_t$, and $\gamma = \sup\{t < T_0 : X_t = 1\}$.*

**Proof of Proposition 6.**
Thanks to (25), the LHS of (26) equals:

$$W_1\left(F_t(1 - X_{t\wedge T_0})^+ 1_{(t \leq T_0)}\right)$$

which is equal to the RHS of (26) thanks to formula (3), applied with $F_t' = F_t\, 1_{(t \leq T_0)}$.

∎

**Remarks.**
**a)** It is worth noting that, as a consequence of (26) there is the identity

$$E_1^{(3)}\left[\left(\frac{1}{X_t} - 1\right)^+\right] = W_1(\gamma \leq t \leq T_0)$$

which shows clearly that the RHS is not an increasing function of $t$; in fact, we can compute explicitly this RHS, which equals:

$$r(t) \equiv W_1(T_0 \geq t) - W_1(\gamma \geq t)$$

Recall that, under $W_1$: $T_0 \overset{\text{(law)}}{=} \frac{1}{B_1^2}$, and $\gamma \overset{\text{(law)}}{=} \frac{U_{[0,2]}}{B_1^2}$ thus:

$$\begin{aligned}
r(t) &= P\left(|B_1| \le \frac{1}{\sqrt{t}}\right) - P\left(|B_1| \le \frac{\sqrt{U_{[0,2]}}}{\sqrt{t}}\right) \\
&= \sqrt{\frac{2}{\pi}} \int_0^{\frac{1}{\sqrt{t}}} dx\, e^{-\frac{x^2}{2}} - \sqrt{\frac{2}{\pi}} \int_0^{\infty} dx\, e^{-\frac{x^2}{2}} \left(1 - \frac{tx^2}{2}\right)^+ \\
&= \sqrt{\frac{2}{\pi}} \int_0^{\frac{1}{\sqrt{t}}} dx\, e^{-\frac{x^2}{2}} \left(\frac{tx^2}{2}\right) - \sqrt{\frac{2}{\pi}} \int_{\frac{1}{\sqrt{t}}}^{\sqrt{\frac{2}{t}}} dx\, e^{-\frac{x^2}{2}} \left(1 - \frac{tx^2}{2}\right)
\end{aligned}$$

**b)** There is an extension of formula (26) for pairs of Bessel processes with respective dimensions $\delta \in (0,2)$, and $4-\delta$; formula (25) generalizes as:

$$P_1^{(4-\delta)}\left[F_t\left(\frac{1}{X_t^{2-\delta}} - 1\right)^+\right] = P_1^{(\delta)}\left(F_t\, 1(\gamma \le t \le T_0)\right)$$

for every $F_t$ in $\mathcal{F}_t$, and $\gamma = \sup\{t < T_0 : X_t = 1\}$.
Proposition 6 leads us easily to a general statement, whose proof simply mimicks that of Proposition 6, hence it is left to the reader.

**Proposition 7.** *Let* $Q_{|\mathcal{F}_t} = M_t \bullet P_{|\mathcal{F}_t}$,
*with* $M_0 = 1$, *and* $T_0 = \inf\{t : M_t = 0\} < \infty$ $P$ *a.s.*
*Then,* $M'_t = \dfrac{1_{(t<T_0)}}{M_t}$ *is well defined and strictly positive under* $Q$*;* $M'_t \in \mathcal{L}_0^{(+)}$*; finally:*

$$E_Q[F_t(M'_t - 1)^+] = E_P[F_t\, 1_{(\gamma \le t < T_0)}] \tag{27}$$

*where:* $\gamma = \sup\{t < T_0 : M_t = 1\}$.

**Remarks.**
**1)** Note that, as for Proposition 6, the RHS of (27), for $F_t \equiv 1$, is no longer an increasing function of $t$.
**2)** The result (27) should be compared with the general expression given in Madan-Yor [4], Proposition 2, p. 160, for $E_Q[(M'_t - K)^+]$. However, we postpone a detailed discussion to another publication.

# 7 Conclusion

• In this Note, we have seen two instances of the following equality:

$$E_P[F_t\, X_t] = \mathcal{Q}(F_t\, 1_{g \le t}) \tag{28}$$

where on the LHS, $X_t \geq 0$, and $F_t$ is any set in $\mathcal{F}_t$, and on the LHS, $\mathcal{Q}$ is a $\geq 0$ finite measure, and $g$ a positive random time.
These two instances are:
**i)** in Theorem 1, $X_t = (b - M_t)^+$,

$$\mathcal{Q} = bP\ ,\ g = g_\infty^{(b)}$$

**ii)** in Proposition 4, $X_t = (M_t - 1)^+\ ,\ \mathcal{Q} = P^M$, and also: $X'_t = |M_t - 1|$, and $\mathcal{Q}' = P + P^M$.
• We are interested in discussing / establishing a general family of identities such as (28), even in cases where $\mathcal{Q}$ is only $\sigma$-finite.
First, note that, if (28) holds, then $(X_t)$ is a $(P, (\mathcal{F}_t))$ submartingale, since, from (28), we get, for $F_s \in \mathcal{F}_s$, and $s \leq t$:

$$E_P(F_s(X_t - X_s)) = \mathcal{Q}(F_s\, 1_{(s<g\leq t)}) \geq 0 \tag{29}$$

Thus, this raises the following question:
for which positive submartingales $(X_t)$ can we exhibit a pair $(\mathcal{Q}, g)$ such that (28) is satisfied?
We do not know the answer to this question in all generality , but we mention two other instances for which (28) is satisfied:
**iii)** (see [6]) In this CRAS Note, and in related papers, one associates to the standard Wiener measure $W$ on canonical space $C(\mathbb{R}_+, \mathbb{R})$ a $\sigma$-finite measure $\mathcal{W}$ such that:

$$W(F_t|X_t|) = \mathcal{W}(F_t\, 1_{(g\leq t)}) \tag{30}$$

for every $F_t$ in $\mathcal{F}_t$, and $g = \sup\{s : X_s = 0\}$
**iv)** (see [1] and [2]) These papers deal with a continuous uniformly integrable martingale $M_t$, with, say, $M_0 = 0$, on a general filtered probability space. The following identity is shown:

$$E_P(F_t|M_t|) = E_P(F_t|M_\infty|1_{(g\leq t)})\ , \tag{31}$$

where $g = \sup\{s : M_s = 0\}$; replacing $M$ by $(M - a)$, and integrating over $a$, (31) is exploited in [1]to present, e.g., the increasing process $\langle M\rangle$of $M$ as a "dual predictable projection". Thus, (31) is another instance of (28), with $\mathcal{Q} = |M_\infty|\bullet P$; clearly, (31) is equivalent to:

$$E_P(|M_\infty|1_{(g\leq t)}|\mathcal{F}_t) = |M_t| \tag{32}$$

Thus, we find - and this may seem a little paradoxical - that in our present framework which involves martingales in $\mathcal{L}_0^{(+)}$, formulae such as (28) are "the simplest possible" in that $\mathcal{Q}$ may be taken to be a multiple of $P$.
We hope, in a further publication, to obtain a unified framework under which formula (28) holds for a large class of positive submartingales.

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Added in Proof (February 2008):
a) A more Mathematical Finance oriented paper has now been written by the authors.
D. Madan, B. Roynette and M. Yor: Option prices as probabilities. To appear in Finance Research Letters (2008).
b) More detailed Notes from the Course of the third author given at the Bachelier Séminaire at IHP (February 8-15-22) are now gathered in the document: "From Black-Scholes and Dupire formulae to last passage times of local martingales", written by A. Bentata and M. Yor.